**Differentiation & Integration – Notes**

axn = anxn+1

dx =

Integrating the derivative of a function “gives the function back” and differentiating the integral of a function “gives the function back”.

( = f(x)

f(x)) = f(x)

> 0 → Positive → Minimum turning point

< 0 → Negative → Maximum turning point

Minimum turning point:

|  |  |  |  |
| --- | --- | --- | --- |
| x | a- | a | a+ |
| y’ | – | 0 | + |

Maximum turning point:

|  |  |  |  |
| --- | --- | --- | --- |
| x | a- | a | a+ |
| y’ | + | 0 | – |

Horizontal inflection point:

|  |  |  |  |
| --- | --- | --- | --- |
| x | a- | a | a+ |
| y’ | ± | 0 | ± |

Differentiate with  
respect to t

Differentiate with  
respect to t

Displacement Velocity Acceleration

Integrate with  
respect to t

Integrate with  
respect to t

Distance = dt

Displacement = dt

Distance → absolute value.

Displacement → not absolute value.

**A curve has equation y = x4 – x2 – 4. Use a calculus method to find the coordinates of all the stationary points on this curve. Use the sign test to determine the nature of these points.**

y’ = 4x3 – 2x → 4x3 – 2x = 0 → CAS solve → x = 0, ±

x = 0 → y = –4 → (0, –4)

|  |  |  |  |
| --- | --- | --- | --- |
| x | 0- | 0 | 0+ |
| x’ | + | 0 | – |

(0, –4) is a maximum turning point.

x = → y = ()4 – ()2 – 4 = → (, )

|  |  |  |  |
| --- | --- | --- | --- |
| x | ()- |  | ()+ |
| x’ | – | 0 | + |

(, ) is a minimum turning point.

x = → y = 4()3 – 2() = 0 → (, 0)

|  |  |  |  |
| --- | --- | --- | --- |
| x | ()- |  | ()+ |
| x’ | – | 0 | + |

(, ) is a minimum turning point.

**A curve has equation y = (x–2)(2x2–5x+2). The points A and B lie on this curve. The tangents to the curve at A and B are parallel to the line 12x – y = 5. Find the coordinates of the points A and B.**

12x – y = 5 → y = 12x – 5 → gradient = 12

y = 2x3 – 9x2 + 12x – 4 → y’ = 6x2 – 18x + 12

6x2 – 18x + 12 = 12 → x = 0, 3

When x = 0, y = –4 → P(A): (0, –4)

When x = 3, y = 5 → P(B): (3, 5)

**The tangent to the curve y = x3(x+2) at the points where x = 1 and x = –1 meet at the point Q. Find the coordinates of the point Q.**

y = x4 + 2x3 → y’ = 4x3 + 6x2 → 4(1)3 + 6(1)2 = 10 → gradient = 10 at x = 1

x = 1 → y = (1)4 + 2(1)3 = 3 → (1, 3) → y = 10x + c → 3 = 10(1) + c → c = –7

y’ = 4x3 + 6x2 → 4(–1)3 + 6(–1)2 = 2 → gradient = 2 when x = –1

x = –1 → (–1)4 + 2(–1)3 = –1 → (–1, –1) → y = 2x + c → –1 = 2(–1) + c → c = 1

y = 10x – 7, y = 2x + 1 → solve simultaneously → 8x – 8 = 0 → x = 1

x = 1 → (1)4 + 2(1)3 = 3 → P(Q): (1, 3)

First, we find the gradients of the tangents at the points in the question, and then we use y = mx + c to find the equations of the tangents. We then solve simultaneously to find ‘x’ at the point of intersection, then substitute to the ‘original’ equation to find ‘y’.

**A curve has equation y = ax3 + bx2 + cx + d. The curve has a turning point at x = 1, a y-intercept at (0, –33) and a tangent with equation y = –24x –37 at x = –1. Find the values of a, b, c and d. Show clearly how you obtained your answer.**

**d = –33**

y’ = 3ax2 + 2bx + c → 3a(1)2 + 2b(1) + c = 0 → 3a + 2b + c = 0

Gradient = –24 at x = –1 → 3a –2b + c = –24

Solve simultaneously → 4b = 24 → **b = 6**

3a + 12 + c = 0 → 3a + c = –12

Substitute (–1, –13) → a(–1)3 + (6)( –1)2 + c(–1) –33 = –13 → –a – c = 14

Solve simultaneously → 2a = 2 → **a = 1**

Substitute (–1, –13) → –13 = –1 + 6 – c –33 → **c = –15**

**Use an appropriate derivative to evaluate ]**

] = |x=5 = |x=5 = = = 1 +

**A closed rectangular box has a volume of 10 000 cm3. The height of the box is twice its width. Use a calculus method to find the dimensions of the box that will minimize its surface area.**

Volume = 10 000 = lwh = lw2w = 2lw2 → 2lw2 = 10 000 → l =

Surface area = 2lw + 2lh + 2wh = 2lw + 4lw + 4w2 = + 4w2

= 8w3 – 30 000 → CAS solve → w = 15.53616253 = 15.54cm

= 24w2 → 24(15.54)2 = 5792.936308 → minimum turning point

l = = 20.71488337 = 20.71cm

h = 2 x 15.54 = 31.07232506 = 31.07cm

**The total surface area of a closed rectangular box is 2000cm2. The length of the box is 4 times its height.**

**[a] Show that the volume of the box is given by V = 800x – 3.2x3.**

Length = l = 4h

Width = w

Height = h

SA = 2000 = 2lw + 2lh + 2wh = 2(4h)w + 2(4h)h + 2wh = 8hw + 8h2 + 2wh

CAS solve for w → w = –0.8h +

Volume = l x w x h = 4h x (–0.8h + ) x h = 4h2 x (–0.8h + ) = –3.2h3 + 800h

= 800x – 3.2x3

**[b] Use a calculus method to find the maximum volume of the box and the corresponding dimensions of the box.**

V’ = 800 – 9.6x2

800 – 9.6x2 = 0 → CAS solve → x = 9.1287...

Height = 9.1cm

Length = 4h = 36.36.5cm

Width = –0.8h + = 14.6cm

**The gradient function of a curve is given by = + 2x – 1. Find the equation of the curve if it passes through (1, 2).**

y = = + x2 – x + c

2 = + (1)2 – 1 + c → c = → y = + x2 – x +

**Find f(x) if f ’(x) = x2 + 2x + k and f(0) = –2 and f(–1) = .**

f(x) = dx = + x2 + kx + c

x = 0 → f(x) = –2 → c = –2

= + (–1)2 + k(–1) – 2 → k = –1

f(x) = + x2 – x – 2

**Find the equation of the tangent(s) to the curve y = – x2 – that are parallel to the line x + y = 6.**

y = –x + 6 → gradient of tangent = –1

y’ = x2 – 2x → x2 – 2x = –1 → x = 1

y = – 12 – = –1

y = –x + c → substitute (1, –1) → –1 = –1 + c → c = 0

Equation: y = –x

**A metal box company is asked to produce cylindrical metal tins, each with a volume of 535cm3. The base and top of each tin have to be made from thicker material than is used for the wall. This thicker material costs twice as much per cm2 as the thinner material. Find, in centimeters and correct to one decimal place, the base radius and height of each tin for the cost of material to be a minimum.**

V = 535 = πr2h → h = =

Total surface area = 2πr2 + 2πrh = 2πr2 + 2πr x = 2πr2 +

Cost = 4πr2 x → = 8πr –

8πr – = 0 → CAS solve → r = 3.49178884 = 3.5cm

= 8π + → = 8π + = 75.39822371 → minimum turning point

h = = 13.96715537 = 14.0cm

radius = 3.5cm, height = 14.0cm

**The displacement of a body moving along a straight line is given by s = –t3 + at2 + bt + 3 metres where t is time in seconds. The initial velocity o the body is 5ms-1. The body is momentarily at rest when t = 1 second.**

**[a] Find the values of a and b.**

= –3t2 + 2at + b → 5 = –3(0)2 + 2a(0) + b → b = 5

–3(1)2 + 2a(1) + 5 = 0 → CAS solve → a = –1

**[b] Find when the body changes direction.**

–3t2 – 2t + 5 = 0 → CAS solve → t = 1

**[c] Find the instantaneous speed at t = 2 seconds and the average speed in the first 2 seconds.**

–3(2)2 – 2(2) + 5 = –11ms-1

Distance = |–t3 + at2 + bt + 3|

When t = 0, distance = 3

When t = 1, distance = 6

When t = 2, distance = 1

Distance travelled = 3 + 5 = 8m

Average speed = 2ms-1

**The velocity v cms-1 of a particle P moving in a straight line at a point x cm from the origin is given by equation v2 = –. P starts from the origin with a velocity of 10cms-1.**

**[a] Show that v2 = 100 – .**

v2 = + c → 100 = 0 + c → c = 100

v2 = 100 –

**[b] Find where P is instantaneously at rest.**

100 – = 0 → CAS solve → x = cm

**[c] Find the maximum speed of P and state where it occurs.**

When x = 0, v2 = 100 → v = 10 → maximum velocity = 10ms-1 at x = 0

**A particle starts off from a fixed point O with an acceleration (mms-2) of a = mt – 24, where t is time in seconds. The particle travels in a straight line and returns to O at t = 4 seconds and has a change of displacement of –9mm in the third second (it moves in the same direction during this time).**

**[a] Find in terms of m an expression for the velocity of the particle at any time t.**

dt =

**[b] Find the displacement of the particle at any time t.**

dt = – + ct + k = – 12t2 + ct + k

t = 0 → – 12(0)2 + c(0) + k = 0 → k = 0

t(3) – t(2) = –9 → – + c(3) – [ – + c(2)] = –9 → + c = 51

– 12(4)2 + c(4) → + 4c = 192

CAS simultaneous → m = 6, c = 32

s = – 12t2 + 32t = t3 – 12t2 + 32t

**[c] Find when the particle is at O the third time (if it does).**

t3 – 12t2 + 32t = 0 → CAS simultaneous → t = 0, 4, 8 → third time is at t = 8 seconds